

On Erdos-Faber-Lovasz Conjecture

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Abstract

In 1972, Erdős - Faber - Lovász (EFL) conjectured that, if \mathbf{H} is a linear hypergraph consisting of n edges of cardinality n , then it is possible to color the vertices with n colors so that no two vertices with the same color are in the same edge. In 1978, Deza, Erdős and Frankl had given an equivalent version of the same for graphs: Let $G = \bigcup_{i=1}^n A_i$ denote a graph with n complete graphs A_1, A_2, \dots, A_n , each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n .

The clique degree $d^K(v)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \leq i \leq n\}|$. In this paper we give a method for assigning colors to the graphs satisfying the hypothesis of the Erds - Faber - Lovsz conjecture using intersection matrix of the cliques A_i 's of G and clique degrees of the vertices of G . Also, we give theoretical proof of the conjecture for some class of graphs. In particular we show that:

1. If G is a graph satisfying the hypothesis of the Conjecture 1.2 and every A_i ($1 \leq i \leq n$) has at most \sqrt{n} vertices of clique degree greater than 1, then G is n -colorable.
2. If G is a graph satisfying the hypothesis of the Conjecture 1.2 and every A_i ($1 \leq i \leq n$) has at most $\lceil \frac{n+d-1}{d} \rceil$ vertices of clique degree greater than or equal to d ($2 \leq d \leq n$), then G is n -colorable.

Keyword *Chromatic number, Erdős - Faber - Lovász conjecture*

MSC[2010] *05A05, 05B15, 05C15*

1 Introduction

One of the famous conjectures in graph theory is Erdős - Faber - Lovász conjecture. It states that if \mathbf{H} is a linear hypergraph consisting of n edges of cardinality n , then it is

possible to color the vertices of \mathbf{H} with n colors so that no two vertices with the same color are in the same edge [1]. Erdős, in 1975, offered 50 USD [4, 10] and in 1981, offered 500 USD [3, 7] for the proof or disproof of the conjecture. Kahn [8] showed that the chromatic number of \mathbf{H} is at most $n + o(n)$. Jakson et al. [6] proved that the conjecture is true when the partial hypergraph S of \mathbf{H} determined by the edges of size at least three can be Δ_S -edge-colored and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$. Sanchez - Arrayo [10] proved the conjecture for dense hypergraphs. Faber [5] proves that for fixed degree, there can be only finitely many counterexamples to EFL on this class (both regular and uniform) of hypergraphs.

Conjecture 1.1 *If \mathbf{H} is a linear hypergraph consisting of n edges of cardinality n , then it is possible to color the vertices of \mathbf{H} with n colors so that no two vertices with the same color are in the same edge.*

We consider the equivalent version of the conjecture for graphs given by Deza, Erdős and Frankl in 1978 [2, 10, 7, 9].

Conjecture 1.2 *Let $G = \bigcup_{i=1}^n A_i$ denote a graph with n complete graphs (A_1, A_2, \dots, A_n) , each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n .*

Definition 1.3 *Let $G = \bigcup_{i=1}^n A_i$ denote a graph with n complete graphs A_1, A_2, \dots, A_n , each having exactly n vertices and the property that every pair of complete graphs has at most one common vertex. The clique degree $d^K(G)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \leq i \leq n\}|$. The maximum clique degree $\Delta^K(G)$ of the graph G is given by $\Delta^K(G) = \max_{v \in V(G)} d^K(v)$.*

From the above definition, one can observe that degree of a vertex in hypergraph is same as the clique degree of a vertex in a graph.

2 Coloring of G

Let G be the graph satisfying the hypothesis of Conjecture 1.2. Let \hat{H} be the graph obtained by removing the vertices of clique degree one from graph G . i.e. \hat{H} is the induced subgraph of G having all the common vertices of G .

Theorem 2.1 *If G is a graph satisfying the hypothesis of the Conjecture 1.2 and every A_i ($1 \leq i \leq n$) has at most \sqrt{n} vertices of clique degree greater than 1, then G is n -colorable.*

Proof: Let G be a graph satisfying the hypothesis of the Conjecture 1.2 and every A_i ($1 \leq i \leq n$) has at most \sqrt{n} vertices of clique degree greater than 1. Let \hat{H} be the induced subgraph of G consisting of the vertices of clique degree greater than one in G . Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \dots, n$.

From [10], it is true that the vertices of clique degree greater than or equal to \sqrt{n} can be assigned with at most n colors. Assign the colors to the vertices of clique degree in non increasing order. Assume we next color a vertex v of clique degree $1 < k < \sqrt{n}$. At this point only vertices of clique degree $\geq k$ have been assigned colors. By assumption every A_i ($1 \leq i \leq n$) has at most \sqrt{n} vertices of clique degree greater than 1 and clique degree of v is k ($k < \sqrt{n}$), then for these k A_i 's there are at most $k\sqrt{n} < n$ vertices have been colored. Therefore, there is an unused color from the set of n colors, then that color can be assigned to the vertex v . ■

Theorem 2.2 *If G is a graph satisfying the hypothesis of the Conjecture 1.2 and every A_i ($1 \leq i \leq n$) has at most $\lceil \frac{n+d-1}{d} \rceil$ vertices of clique degree greater than or equal to d ($2 \leq d \leq n$), then G is n -colorable.*

Proof: Let G be a graph satisfying the hypothesis of the Conjecture 1.2 and every A_i ($1 \leq i \leq n$) has at most $\lceil \frac{n}{d} \rceil$ vertices of clique degree greater than or equal to d ($2 \leq d \leq n$). Let \hat{H} be the induced subgraph of G consisting of the vertices of clique degree greater than one in G . Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \dots, m$.

Assign the colors to the vertices of clique degree in non increasing order. Assume we next color a vertex v of clique degree k . At this point only vertices of clique degree $\geq k$ have been assigned colors. By assumption every A_i ($1 \leq i \leq n$) has at most \sqrt{n} vertices of clique degree greater than 1 and clique degree of v is k ($k < \sqrt{n}$), then for these k A_i 's there are at most $k(\lceil \frac{n}{k} \rceil - 1) < n$ vertices have been colored. Therefore, there is an unused color from the set of n colors, then that color can be assigned to the vertex v . ■

Let G be a graph satisfying the hypothesis of the Conjecture 1.2, the intersection matrix is a square $n \times n$ matrix C such that its element $c_{i,j}$ is c when $A_i \cap A_j \neq \emptyset$ otherwise zero for $i \neq j$ and the diagonal elements of the matrix are all zero.

Given below is a method to color graph G satisfying the hypothesis of the Conjecture 1.2 using intersection matrix (color matrix) of the cliques A_i 's of G and clique degrees of the vertices of G .

Method for assigning colors to graph G :

Let G be a graph satisfying the hypothesis of Conjecture 1.2. Let \hat{H} be the induced subgraph of G consisting of the vertices of clique degree greater than 1 in G . Relabel the vertex v of clique degree greater than 1 in G by u_x , where $x = k_1, k_2, \dots, k_j$; vertex v is in A_{k_i} , $1 \leq i \leq j$. Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \dots, n$.

Let C be the intersection matrix of the cliques A_i 's of G where $c_{i,j} = 0$ if $A_i \cap A_j = \emptyset$ otherwise c for $i \neq j$ and $c_{i,i}$ is 0. Let $1, 2, \dots, n$ be the n -colors.

Let $T_i = X_i$, $P_i = \emptyset$ and $S = \{j : T_j \neq \emptyset, 2 \leq j \leq n\}$.

If $S = \emptyset$, then the graph G has no vertex of clique degree greater than one, which implies G has exactly n^2 (maximum number) vertices. i.e., G is n components of K_n . Otherwise, consider the intersection matrix C as the color matrix and follow the steps.

Step 1: If $S = \emptyset$, stop the process. Otherwise, let $\max(S) = k$, for some $k, 2 \leq k \leq n$. Then consider the sets T_k and P_k , go to step 2.

Step 2: If $T_k = \emptyset$, go to step 1. Otherwise, choose a vertex u_{i_1, i_2, \dots, i_k} from T_k , where $i_1 < i_2 < \dots < i_k$ and go to Step 3.

Step 3: Let $Y_i = \{y : \text{color } y \text{ appears atleast } k-1 \text{ times in the } i^{\text{th}} \text{ row of the color matrix}\}$, $i = 1, 2, \dots, n$. If $|\bigcup_{i=i_1}^{i_k} Y_i| = n$, let $B_T = \bigcup_{i=2}^n P_i$, $B_P = \emptyset$ and go to Step 4. Otherwise, construct a new color matrix C_1 by putting least x in $c_{i,j}$, where $x \in \{1, 2, 3, \dots, n\} \setminus \bigcup_{i=i_1}^{i_k} Y_i$, $i \neq j$, $i_1 \leq i, j \leq i_k$. Then add the vertex u_{i_1, i_2, \dots, i_k} to P_k and remove it from T_k , go to Step 2.

Step 4: Choose a vertex v from B_T such that $v \in A_i$, for some i , $i_1 \leq i \leq i_k$. Let $B = \{i : v \in A_i, 1 \leq i \leq n\}$ and go to Step 5.

Step 5: Let $Y_i = \{y : \text{color } y \text{ appears atleast } k-1 \text{ times in the } i^{\text{th}} \text{ row of the color matrix}\}$, for every $i \in B$. If $|\bigcup_{i \in B} Y_i| = n$ add the vertex v to B_P and remove it from B_T , go to Step 4. Otherwise construct a new color matrix C_2 by putting x in $c_{i,j}$, where $x \in \{1, 2, 3, \dots, n\} \setminus \bigcup_{i \in B} Y_i$, $i \neq j$, $i, j \in B$. Go to Step 3.

Thus, we get the modified color matrix C_M . Then, color the vertex v of \hat{H} by $c_{i,j}$ of C_M , whenever $v \in A_i \cap A_j$. Then, extend the coloring of \hat{H} to G . Thus G is n -colorable.

Following is an example illustrating the above method.

Example 2.3 Let G be the graph shown in Figure 1a.

Let $V(A_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V(A_2) = \{v_1, v_7, v_8, v_9, v_{10}, v_{11}\}$,
 $V(A_3) = \{v_1, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, $V(A_4) = \{v_1, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}\}$,
 $V(A_5) = \{v_6, v_7, v_{16}, v_{22}, v_{23}, v_{24}\}$, $V(A_6) = \{v_9, v_{16}, v_{19}, v_{25}, v_{26}, v_{27}\}$.

Relabel the vertices of clique degree greater than one in G by u_A where $A = \{i : v \in A_i \text{ for } 1 \leq i \leq 6\}$. The labeled graph is shown in Figure 1b. Figure 2 is the graph \hat{H} , where \hat{H} is obtained by removing the vertices of clique degree 1 from G .

Let $X = \{b_{i,j} : A_i \cap A_j = \emptyset\} = \{b_{1,6}, b_{4,5}\}$,

$$X_1 = \{v \in G : d^K(v) = 1\} = \{v_2, v_3, v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, \\ v_{17}, v_{18}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\},$$

$$X_2 = \{v \in G : d^K(v) = 2\} = \{v_6, v_7, v_9, v_{19}\} = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\},$$

$$X_3 = \{v \in G : d^K(v) = 3\} = \{v_{16}\} = \{u_{3,5,6}\},$$

$$X_4 = \{v \in G : d^K(v) = 4\} = \{v_1\} = \{u_{1,2,3,4}\},$$

$$X_5 = \emptyset \text{ and } X_6 = \emptyset.$$

$$\text{Let } 1, 2, \dots, 6 \text{ be the six colors and } C = \begin{pmatrix} 0 & c & c & c & c & 0 \\ c & 0 & c & c & c & c \\ c & c & 0 & c & c & c \\ c & c & c & 0 & 0 & c \\ c & c & c & 0 & 0 & c \\ 0 & c & c & c & c & 0 \end{pmatrix}$$

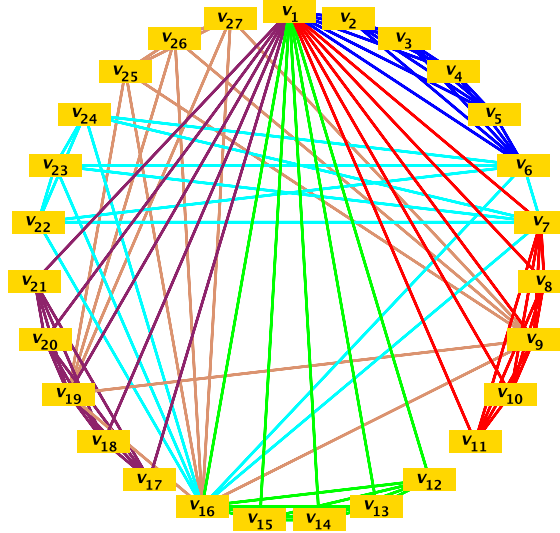
be the intersection matrix of order 6×6 .

Consider the sets $T_i = X_i$, $P_i = \emptyset$ for $i = 1, 2, \dots, 6$ and $S = \{j : T_j \neq \emptyset, 2 \leq j \leq n\} = \{2, 3, 4\}$. Then by applying the method given above, we get the following.

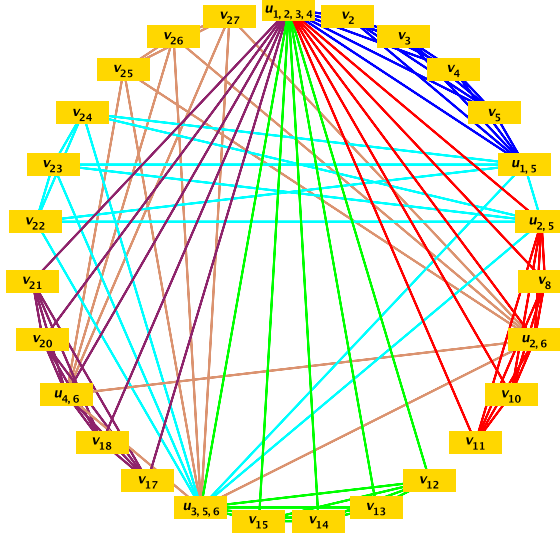
Step 1: Since $S \neq \emptyset$ and $\max(S) = 4$, then choose the sets $T_4 = \{u_{1,2,3,4}\}$ and $P_4 = \emptyset$. Go to step 2.

Step 2: Since $T_4 \neq \emptyset$, choose the vertex $u_{1,2,3,4}$ from T_4 , go to step 3.

Step 3: Since $Y_1 = \emptyset$, $Y_2 = \emptyset$, $Y_3 = \emptyset$, $Y_4 = \emptyset$ and $|Y_1 \cup Y_2 \cup Y_3 \cup Y_4| < 6$, choose the minimum color from the set $\{1, 2, \dots, 6\} \setminus \cup_{i=1,2,3,4} Y_i$ and construct a new color



(a) Graph G



(b) Graph G after relabeling the vertices of clique degree greater than one

Figure 1: Graph G : before and after relabeling the vertices

matrix C_1 by putting 1 in $c_{i,j}$, $i \neq j$, $i, j = 1, 2, 3, 4$. Add the vertex $u_{1,2,3,4}$ to P_4 and remove it from T_4 . Then

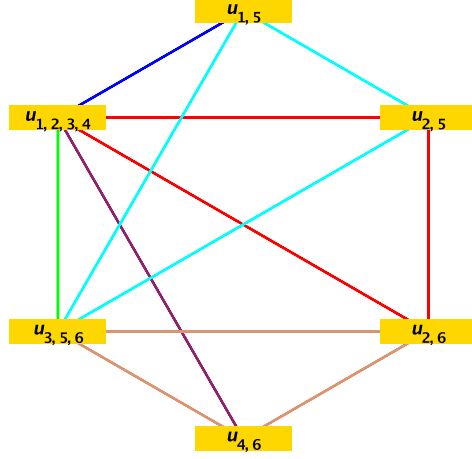


Figure 2: Graph \hat{H}

$$C_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & c & 0 \\ 1 & 0 & 1 & 1 & c & c \\ 1 & 1 & 0 & 1 & c & c \\ 1 & 1 & 1 & 0 & 0 & c \\ c & c & c & 0 & 0 & c \\ 0 & c & c & c & c & 0 \end{pmatrix},$$

$T_4 = \emptyset$, $P_4 = \{u_{1,2,3,4}\}$. Go to step 2.

Step 2: Since $T_4 = \emptyset$, go to step 1.

Step 1: Since $S \neq \emptyset$ and $\max(S) = 3$, then choose the sets $T_3 = \{u_{3,5,6}\}$ and $P_3 = \emptyset$. Go to step 2.

Step 2: Since $T_3 \neq \emptyset$, choose the vertex $u_{3,5,6}$ from T_3 , go to step 3.

Step 3: Since $Y_3 = \{1\}$, $Y_5 = \emptyset$, $Y_6 = \emptyset$, and $|Y_3 \cup Y_5 \cup Y_6| < 6$, choose the minimum color from the set $\{1, 2, \dots, 6\} \setminus \cup_{i=3,5,6} Y_i$ and construct a new color matrix C_2 by putting 2 in $c_{i,j}$, $i \neq j$, $i, j = 3, 5, 6$. Add the vertex $u_{3,5,6}$ to P_3 and remove it from T_3 . Then

$$C_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & c & 0 \\ 1 & 0 & 1 & 1 & c & c \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ c & c & 2 & 0 & 0 & 2 \\ 0 & c & 2 & c & 2 & 0 \end{pmatrix},$$

$T_3 = \emptyset$, $P_3 = \{u_{3,5,6}\}$. Go to step 2.

Step 2: Since $T_3 = \emptyset$, go to step 1.

Step 1: Since $S \neq \emptyset$ and $\max(S) = 2$, then choose the sets $T_2 = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$ and $P_2 = \emptyset$. Go to step 2.

Step 2: Since $T_2 \neq \emptyset$, choose the vertex $u_{1,5}$ from T_2 , go to step 3.

Step 3: Since $Y_1 = \{1\}$, $Y_5 = \{2\}$ and $|Y_1 \cup Y_5| < 6$, choose the minimum color from the set $\{1, 2, \dots, 6\} \setminus \cup_{i=1,5} Y_i$ and construct a new color matrix C_3 by putting 3 in $c_{i,j}$, $i \neq j$, $i, j = 1, 5$. Add the vertex $u_{1,5}$ to P_2 and remove it from T_2 . Then

$$C_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & c & c \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ 3 & c & 2 & 0 & 0 & 2 \\ 0 & c & 2 & c & 2 & 0 \end{pmatrix},$$

$T_2 = \{u_{2,5}, u_{2,6}, u_{4,6}\}$, $P_2 = \{u_{1,5}\}$. Go to step 2.

Step 2: Since $T_2 \neq \emptyset$, choose the vertex $u_{2,5}$ from T_2 , go to step 3.

Step 3: Since $Y_2 = \{1\}$, $Y_5 = \{2, 3\}$ and $|Y_2 \cup Y_5| < 6$, choose the minimum color from the set $\{1, 2, \dots, 6\} \setminus \cup_{i=2,5} Y_i$ and construct a new color matrix C_4 by putting 4 in $c_{i,j}$, $i \neq j$, $i, j = 2, 5$. Add the vertex $u_{2,5}$ to P_2 and remove it from T_2 . Then

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & 4 & c \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ 3 & 4 & 2 & 0 & 0 & 2 \\ 0 & c & 2 & c & 2 & 0 \end{pmatrix},$$

$T_2 = \{u_{2,6}, u_{4,6}\}$, $P_2 = \{u_{1,5}, u_{2,5}\}$. Go to step 2.

Step 2: Since $T_2 \neq \emptyset$, choose the vertex $u_{2,6}$ from T_2 , go to step 3.

Step 3: Since $Y_2 = \{1, 4\}$, $Y_6 = \{2\}$ and $|Y_2 \cup Y_6| < 6$, choose the minimum color from the set $\{1, 2, \dots, 6\} \setminus \cup_{i=2,6} Y_i$ and construct a new color matrix C_5 by putting 3 in $c_{i,j}$, $i \neq j$, $i, j = 2, 6$. Add the vertex $u_{2,6}$ to P_2 and remove it from T_2 . Then

$$C_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & 4 & 3 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ 3 & 4 & 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & c & 2 & 0 \end{pmatrix},$$

$T_2 = \{u_{4,6}\}$, $P_2 = \{u_{1,5}, u_{2,5}, u_{2,6}\}$. Go to step 2.

Step 2: Since $T_2 \neq \emptyset$, choose the vertex $u_{4,6}$ from T_2 , go to step 3.

Step 3: Since $Y_4 = \{1\}$, $Y_6 = \{2, 3\}$ and $|Y_4 \cup Y_6| < 6$, choose the minimum color from the set $\{1, 2, \dots, 6\} \setminus \cup_{i=4,6} Y_i$ and construct a new color matrix C_6 by putting 4 in $c_{i,j}$, $i \neq j$, $i, j = 4, 6$. Add the vertex $u_{4,6}$ to P_2 and remove it from T_2 . Then

$$C_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & 4 & 3 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 3 & 4 & 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & 4 & 2 & 0 \end{pmatrix},$$

$T_2 = \emptyset$, $P_2 = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$. Go to step 2.

Step 2: Since $T_2 = \emptyset$, go to step 1.

Step 1: Since $S = \emptyset$, stop the process.

Assign the colors to the graph \hat{H} using the matrix $C_M = C_6$, i.e., color the vertex v by the (i, j) -th entry $c_{i,j}$ of the matrix C_M , whenever $A_i \cap A_j \neq \emptyset$ (see Figure 3a), where the numbers 1, 2, 3, 4, 5, 6 corresponds to the colors Maroon, Tan, Green, Red, Blue, Cyan respectively. Extend the coloring of \hat{H} to G by assigning the remaining colors which are not used for A_i from the set of 6-colors to the vertices of clique degree one in each A_i , $1 \leq i \leq 6$. The colored graph G is shown in Figure 3b.

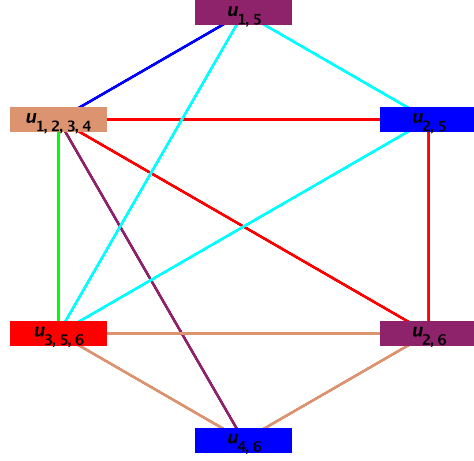
The following results give a relation between the number of complete graphs and clique degrees of a graph.

Theorem 2.4 *Let G be a graph satisfying the hypothesis of Conjecture 1.2. Then if the intersection of any two A_i 's is non empty, then*

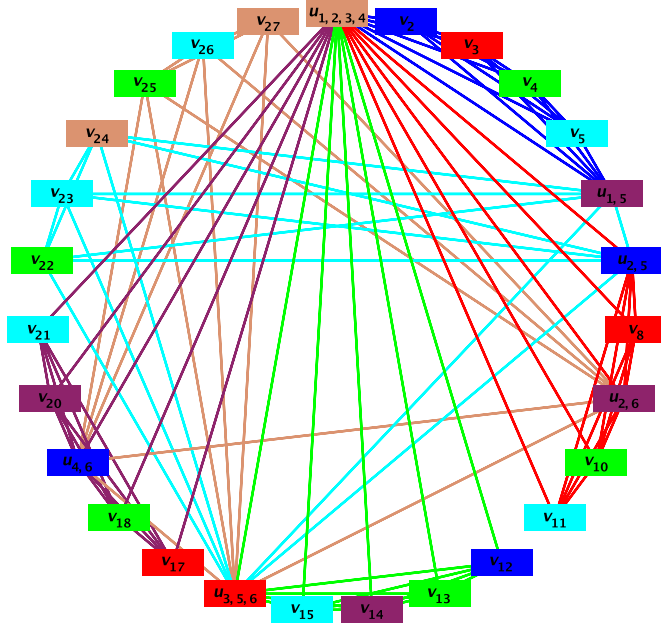
$$\binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \dots + \binom{d^K(v_l)}{2} = \frac{n(n-1)}{2},$$

where $\{v_1, v_2, \dots, v_l\}$ is the set of all vertices of clique degree greater than 1 in G .

Proof: If G is isomorphic to the graph H_n for some n , then the result is obvious. If not there exists at least one vertex v of clique degree greater than 2. Define $I_v = \{i : v \in A_i\}$ then $d^K(v) = |I_v| = p$. For every unordered pair of elements (i, j) of I_v there is a vertex b_{ij} (where $i < j$) in H_n . Therefore corresponding to the elements of I_v there are $\binom{p}{2}$ vertices in H_n . Since G satisfies the hypothesis of



(a) Graph \hat{H}



(b) A 6 coloring of graph G

Figure 3: The graphs \hat{H} and G , after colors have been assigned to their vertices.

Conjecture 1.2, there is no vertex v' different from v in G such that $v' \in A_i \cap A_j$ where $i, j \in I_v$. Therefore for every vertex v of clique degree greater than 1 in G , there are $\binom{d^K(v)}{2}$ vertices of clique degree greater than 1 in H_n . As there are $\frac{n(n-1)}{2}$ vertices of clique degree greater than 1 in H_n , $\frac{n(n-1)}{2} = \binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \dots + \binom{d^K(v_l)}{2}$ where $\{v_1, v_2, \dots, v_l\}$ is the set of all vertices of clique degree greater than 1 in G . ■

Corollary 2.5 *If G is a graph satisfying the hypothesis of conjecture 1.2, then G has*

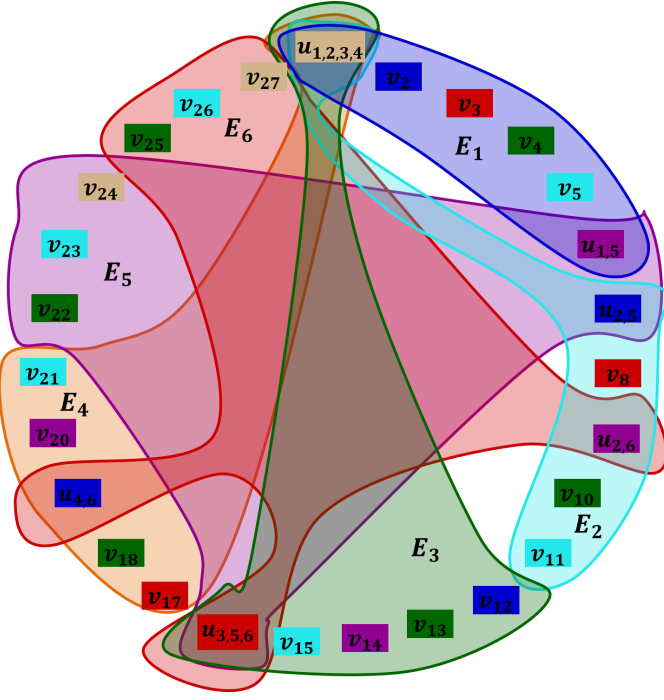


Figure 4: A 6 coloring of hypergraph \mathbf{H} corresponding to the graph G shown in Figure 3b

at most $\frac{\binom{n}{2}}{\binom{m}{2}}$ vertices of clique degree m where $m \geq 2$.

Proof: Let $A = \{v_1, v_2, \dots, v_l\}$ be the set of vertices of clique degree greater than 1 in G and $p = \frac{\binom{n}{2}}{\binom{m}{2}}$. We have to prove that G has at most p vertices of clique degree m . Suppose G has $q > p$ vertices of clique degree m . Then by the definition of A , it follows that, q vertices are in A . Let those vertices be v_1, v_2, \dots, v_q . By Theorem 2.4 we get,

$$\begin{aligned}
 \frac{n(n-1)}{2} &= \binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \dots + \binom{d^K(v_l)}{2} \\
 &\geq \binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \dots + \binom{d^K(v_q)}{2} \\
 &= q \binom{m}{2} \\
 &\geq (p+1) \binom{m}{2} \\
 \frac{\binom{n}{2}}{\binom{m}{2}} &\geq p+1 \\
 p &\geq p+1,
 \end{aligned}$$

which is a contradiction. Hence there are at most $\frac{\binom{n}{2}}{\binom{m}{2}}$ vertices of clique degree m in G , where $m \geq 2$. ■

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